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Stable Coupling Method for Interface Scattering Problems by Combined Integral Equations and Finite Elements

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The problem of elastic-fluid scattering is studied by combined integral equations for the exterior acoustic fluid and finite element methods for the elastic structure. These yield reduced problems over finite domains. The well-known difficulty with exterior integral equation methods at critical frequencies carries over to the reduced problem. A procedure is given to overcome this difficulty and a complete analysis of this procedure is provided, showing existence, uniqueness, and optimal convergence for the coupled problems. This method is termed stable. Numerical experiments confirm that the new approach is valid for all frequencies and indicate that variational methods are preferable to collocation in the treatment of integral equations, even for purely exterior problems. Our method is fully variational and may be expressed in a symmetric form. © 1995 Academic Press, Inc.

I. INTRODUCTION

In this paper we are concerned with the problem of scattering by an elastic obstacle in an acoustic fluid. This is intended as a model for a class of scattering problems and their solution by what are known as coupling methods. We discuss both questions of existence, uniqueness, and convergence, as well as issues of accuracy of the numerical approximations. Let us state the mathematical problem. Physical details appear in [1, 7]. We describe the three-dimensional problem but our computations will be done in two dimensions.

Let Ω be a bounded region in $\mathbf{R}^3$, with boundary Γ and exterior Ω^+. Let $\boldsymbol{\sigma}[\mathbf{U}]$ denote the stress tensor for a linearly elastic, inhomogeneous material for displacement $\mathbf{U}$. ρ is a positive function in $\overline{\Omega}$ and ρ_0, c_0, and ω are positive constants. p^0 is a solution of $\Delta p^0 + (\omega^2/c_0^2)p^0 = 0$ in $\mathbf{R}^3$. Then we seek $\mathbf{U}(\mathbf{x})$ in Ω and $p(\mathbf{x})$ in Ω^+ such that

$$\operatorname{div} \boldsymbol{\sigma}[\mathbf{U}] + \rho\omega^2\mathbf{U} = 0 \quad \text{in } \Omega$$

$$L_\omega p = \Delta p + \frac{\omega^2}{c_0^2} p = 0 \quad \text{in } \Omega^+$$

$$\boldsymbol{\sigma}[\mathbf{U}^-](\mathbf{n}) = -(p^+ + p^0)\mathbf{n} \quad \text{on } \Gamma \qquad \text{(P)}$$

$$\mathbf{U}^- \cdot \mathbf{n} = \frac{1}{\rho_0\omega^2}(p_n^+ + p_n^0) \quad \text{on } \Gamma$$

p satisfies a radiation condition in Ω^+.

$\mathbf{n}$ is the exterior normal to Γ, p_n is the normal derivative of p, and the plus and minus signs indicate limits from Ω^+ and Ω. Ω represents an inhomogeneous elastic obstacle and Ω^+ is a compressible, non-viscous, homogeneous fluid. $\mathbf{U}$ is displacement and $P = p + p^0$ is fluid pressure with p^0 an incident field. All fields are assumed small and time periodic with frequency ω. Analogous problems occur in elastic–elastic and electromagnetic scattering. See, for instance, [2, 5, 10, 11, 13].

Remark 1.1. It is known that, for very special geometry Ω, uniqueness for (P) can fail. We assume here that (P) *has at most one solution.*

Coupling methods[1] for (P) (and analogous problems) proceed as follows. One uses known integral equation methods for exterior problems for $L_\omega p = 0$ to obtain a reduced problem $(\tilde{P})$ over Ω only but with auxiliary boundary functions and nonlocal boundary conditions. One then utilizes standard interior procedures to obtain a variational problem $(V\tilde{P})$ and implements this with finite elements to obtain approximate problems $(AV\tilde{P})$ which are solved numerically. Suppose that, for all frequencies ω, (P) and $(\tilde{P})$ are equivalent, $(\tilde{P})$ has a unique solution, and one has optimal convergence for $(AV\tilde{P})$. Then we say that the coupling method is *stable*.

In Section II we will present three distinct coupling methods. These are derived from three different representations of solutions of $L_\omega p = 0$ in Ω^+. The first two are ones given in [1, 8] and each fails for (different) sets of frequencies. This reflects the fact that the first (second) representations will not succeed as an ansatz to solve the exterior Dirichlet (Neumann) problem for $L_\omega p = 0$. There have been a number of methods proposed to correct their difficulties; see, for instance, [3, 4, 14, 15]. The

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[1] Other types of coupling which do not use integral equations have also been used.

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third coupling method is a stable one. In it we use the exterior representations used in [3]. It is shown in [3] that this new representation will yield a solution of either the Dirichlet or Neumann exterior problem. We show here that it also works for the interface problem. Thus, we may consider our method to give exact results up to the limits of numerical accuracy.

A complete theory is carried out for our stable method. In Section III we derive the variational problem $(V\tilde{P})$ and its approximation $(AV\tilde{P})$. For those interested in technical detail, Section IV contains proofs of existence and optimal convergence. Section VI contains some numerical examples.

Our stable method has essentially the same computational complexity as non-stable variational ones and is comparable to widely used non-stable collocation techniques. For this reason we have included some computations with the non-stable ones, using both collocation and Galerkin techniques; results of these computations suggest two conclusions which seem important and are included in the following remarks.

Remark 1.2. If a non-stable representation is used to solve a pure exterior problem it appears to be preferable to use variational methods on the integral equations rather than collocation. With variational methods errors are highly localized, in contrast to collocation methods. This is dramatically illustrated in Fig. 2. The coupling methods of [1, 8] automatically give variational treatments.

Remark 1.3. Our calculations indicate that non-stable variational methods yield results for the exterior problem which are quite accurate unless ω is essentially equal to a breakdown frequency. The same comment usually holds for the interface problem, although some differences were observed for values of ω near some of the actual natural frequencies which occur in the interface problem.

Remark 1.4. Our method, in principle, will apply to any linear, time-periodic scattering problem in which there are representations of the exterior field in terms of simple and double layer potentials, including electromagnetic, and elastic–elastic problems.

Remark 1.5. One feature of the various coupling methods is the absence or presence of hypersingular integral operators. The examples in Section II illustrate this. We conjecture that stable methods will always contain such operators, although, of course, this is not sufficient. We comment more on this in Section II.

Remark 1.6. A desirable feature for computations on coupling methods is symmetry in $(V\tilde{P})$. In Section V we give a symmetric version of our stable procedure. It contains an additional auxiliary function, but this, however, represents little additional computational effort.

II. THE REDUCED PROBLEMS

The basis for exterior methods is some well-known potential theory. We start with the fundamental singularity for $L_\omega v = 0$ with a radiation condition,[2]

$$K(z) = -\frac{1}{4\pi z}\exp\left(i\frac{\omega}{c_o}z\right) \quad \text{in } \mathbf{R}^3 \tag{2.1}$$

and we form the simple and double layers,

$$\mathcal{S}[\phi](\mathbf{x}) = \int_\Gamma \phi(\mathbf{y})K(|\mathbf{x}-\mathbf{y}|)\,dS_y \tag{2.2a}$$

$$\mathcal{D}[\phi](\mathbf{x}) = \int_\Gamma \phi(\mathbf{y})\frac{\partial}{\partial n_y}K(|\mathbf{x}-\mathbf{y}|)\,dS_y. \tag{2.2b}$$

These satisfy $L_\omega v = 0$ in Ω and Ω^+, with a radiation condition. For smooth ϕ and Γ one has the jump relations

$$\mathcal{S}[\phi]^\pm = S[\phi] \tag{2.3a}$$

$$\mathcal{D}[\phi]^\pm = \mp\tfrac{1}{2}\phi + D[\phi] \tag{2.3b}$$

$$\frac{\partial}{\partial n}\mathcal{S}[\phi]^\pm = \pm\tfrac{1}{2}\phi + N[\phi] \tag{2.3c}$$

$$\frac{\partial}{\partial n}\mathcal{D}[\phi]^\pm = M[\phi] \tag{2.3d}$$

in which

$$D[\phi](\mathbf{x}) = \int_\Gamma \phi(\mathbf{y})\frac{\partial}{\partial n_y}K(|\mathbf{x}-\mathbf{y}|)|_{x\in\Gamma}\,dS_y \tag{2.3e}$$

$$N[\phi](\mathbf{x}) = \int_\Gamma \phi(\mathbf{y})\frac{\partial}{\partial n_x}K(|\mathbf{x}-\mathbf{y}|)|_{x\in\Gamma}\,dS_y \tag{2.3f}$$

$$M[\phi](\mathbf{x}) = \int_\Gamma \phi(\mathbf{y})\frac{\partial^2}{\partial n_x\partial n_y}K(|\mathbf{x}-\mathbf{y}|)|_{x\in\Gamma}\,dS_y. \tag{2.3g}$$

The integral operators D and N have kernels which are of order $|\mathbf{x}-\mathbf{y}|^{-1}$ near $\mathbf{x}=\mathbf{y}$, while M has a kernel of order $|\mathbf{x}-\mathbf{y}|^{-2}$.

Later we will need to consider the integral operators in (2.3) on boundary Sobolev spaces. It is known (see, for instance, [6]) that the following can be defined as boundary linear maps:

$$\begin{aligned} &S, N, D: H_r(\Gamma) \to H_{r+1}(\Gamma),\\ &M: H_{r+1}(\Gamma) \to H_r(\Gamma), \quad r \geq -\tfrac{1}{2}. \end{aligned} \tag{2.4}$$

The last result is what is meant by M being hypersingular. We note that N and D are adjoints while S and M are self-adjoint (with respect to the bilinear form $\langle\phi,\psi\rangle = \int_\Gamma \phi\psi\,ds$).

For a solution of $L_\omega v = 0$ in Ω or in Ω^+ with a radiation condition one has the representation formulas

[2] In $\mathbf{R}^2$ $K(z)$ is replaced by the singular Hankel function $i/4H_0^1(\omega z/c_0)$.

$$v = \mathcal{D}[v^-] - \mathcal{S}[v_n^-] \quad \text{in } \Omega \tag{2.5a}$$

$$v = \mathcal{S}[v_n^+] - \mathcal{D}[v^+] \quad \text{in } \Omega^+. \tag{2.5b}$$

We want to discuss in some detail three different coupling methods. All start with some representation of p in Ω^+. The first is a special case of the method in [1]. It is termed *indirect* since it uses an auxiliary function in the exterior representation. The second comes from [8] and is called *direct* since it uses (2.5b). Each has its advantages and disadvantages but both lack stability. The third method, which is the main topic of this paper, is also indirect but it is stable.

METHOD 1. $p = \mathcal{S}[\phi]$ in Ω^+. (2.3) yields $p^+ = S[\phi]$, $p_n^+ = \frac{1}{2}\phi + N[\phi]$. We substitute these into the boundary conditions and obtain the reduced problem:

$$\begin{aligned} &\operatorname{div} \boldsymbol{\sigma}[\mathbf{U}] + \rho\omega^2\mathbf{U} = 0 && \text{in } \Omega \\ &\boldsymbol{\sigma}[\mathbf{U}^-](\mathbf{n}) = -(S[\phi] + p^0)\mathbf{n} && \text{on } \Gamma \\ &\rho_0\omega^2\mathbf{U}^- \cdot \mathbf{n} = \tfrac{1}{2}\phi + N[\phi] + p_n^0 && \text{on } \Gamma. \end{aligned} \tag{$\tilde{P}_1$}$$

We solve this for $(\mathbf{U}, \phi)$ and determine p in Ω^+ from $p = \mathcal{S}[\phi]$.

METHOD 2. $p = \mathcal{S}[p_n^+] - \mathcal{D}[p^+]$ in Ω^+. (2.3) gives $\frac{1}{2}p^+ = S[p_n^+] - D[p^+]$, $\frac{1}{2}p_n^+ = N[p_n^+] - M[p^+]$. This time we use $\phi = p^+$ as an auxiliary variable and write

$$\begin{aligned} \boldsymbol{\sigma}[\mathbf{U}^-](\mathbf{n}) &= -(p^+ + p^0)\mathbf{n} = -\left(\frac{1}{2}\phi + \frac{1}{2}p^+ + p^0\right)\mathbf{n} \\ &= -\left(\frac{1}{2}\phi + S[p_n^+] - D[\phi] + p^0\right)\mathbf{n} \\ &= -\left(\frac{1}{2}\phi + \rho_0\omega^2 S[\mathbf{U}^-\cdot\mathbf{n}] - S[p_n^0] - D[\phi] + p^0\right)\mathbf{n} \\ &\quad \text{on } \Gamma \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\mathbf{U}^-\cdot\mathbf{n} &= \frac{1}{2\rho_0\omega^2}(p_n^+ + p_n^0) = \frac{1}{\rho_0\omega^2}\left(N[p_n^+] - M[\phi] + \frac{1}{2}p_n^0\right) \\ &= N[\mathbf{U}^-\cdot\mathbf{n}] - \frac{1}{\rho_0\omega^2}\left(M[\phi] + \frac{1}{2}p_n^0 - N[p_n^0]\right) \\ &\quad \text{on } \Gamma. \end{aligned}$$

The reduced problem is, then,

$$\begin{aligned} &\operatorname{div} \boldsymbol{\sigma}[\mathbf{U}] + \rho\omega^2\mathbf{U} = 0 \quad \text{in } \Omega \\ &\boldsymbol{\sigma}[\mathbf{U}^-](\mathbf{n}) = -\left(\frac{1}{2}\phi + \rho_0\omega^2 S[\mathbf{U}^-\cdot\mathbf{n}] - D[\phi] - S[p_n^0] + p^0\right)\mathbf{n} \quad \text{on } \Gamma \\ &\frac{1}{2}\mathbf{U}^-\cdot\mathbf{n} = N[\mathbf{U}^-\cdot\mathbf{n}] - \frac{1}{\rho_0\omega^2}\left(M[\phi] + \frac{1}{2}p_n^0 - N[p_n^0]\right) \\ &\quad \text{on } \Gamma \end{aligned} \tag{$\tilde{P}_2$}$$

for $(\mathbf{U}, \phi)$. p is then determined in Ω^+ by

$$p = \rho_0\omega^2\mathcal{S}[\mathbf{U}^-\cdot\mathbf{n}] - \mathcal{D}[\phi] - \mathcal{S}[p_n^0]. \tag{2.7}$$

METHOD 3. Following [3] we represent p as:

$$p = \mathcal{S}[\phi] + \alpha\mathcal{D}[\phi], \quad \operatorname{Im} \alpha \neq 0 \quad \text{in } \Omega^+. \tag{I}$$

Here $p^+ = S[\phi] - (\alpha/2)\phi + \alpha D[\phi]$, $p_n^+ = \frac{1}{2}\phi + N[\phi] + \alpha M[\phi]$, and the reduced problem is

$$\begin{aligned} &\operatorname{div} \boldsymbol{\sigma}[\mathbf{U}] + \rho\omega^2\mathbf{U} = 0 \quad \text{in } \Omega \\ &\boldsymbol{\sigma}[\mathbf{U}^-](\mathbf{n}) = -\left(S[\phi] - \frac{\alpha}{2}\phi + \alpha D[\phi] + p^0\right)\mathbf{n} \quad \text{on } \Gamma \\ &\rho_0\omega^2\mathbf{U}^-\cdot\mathbf{n} = \frac{1}{2}\phi + N[\phi] + \alpha M[\phi] + p_n^0 \quad \text{on } \Gamma \end{aligned} \tag{$\tilde{P}$}$$

for $(\mathbf{U}, \phi)$ with p determined by (I).

$(\tilde{P}_2)$ contains the hypersingular operator M while $(\tilde{P}_1)$ does not. On the other hand, the variational form $(V\tilde{P}_2)$ is symmetric while $(V\tilde{P}_1)$ is not. (A symmetric version of $(\tilde{P}_1)$, without M, is given in [1].) As we show shortly all these methods are non-stable while Method 3 is stable.

One verifies easily that if $(\mathbf{U}, \phi)$ is a solution of $(\tilde{P}_1)$, $(\tilde{P}_2)$, or $(\tilde{P})$ and p is appropriately defined in Ω^+, $(\mathbf{U}, p)$ will be a solution of (P). The difficulty lies with the reduced problems. The arguments in the next sections can be applied to all three methods to show that one has existence for the reduced problems if and only if one has uniqueness, and uniqueness holds for all ω only for $(\tilde{P})$. Let us make this precise. We have the following result.

LEMMA 2.1. (i) *$\mathcal{S}[\phi] \equiv 0$ in Ω^+ ($\mathcal{D}[\phi] \equiv 0$ in Ω^+) if and only if $\phi = -v_n^-$, where $L_\omega v = 0$ in Ω, $v^- = 0$ ($\phi = -v^-$, where $L_\omega v = 0$ in Ω, $v_n^- = 0$). (ii) If $\operatorname{Im} \alpha \neq 0$ $\mathcal{S}[\phi] + \alpha\mathcal{D}[\phi] \equiv 0$ in $\Omega^+ \Rightarrow \phi \equiv 0$.*

We prove this result at the end of the section but first let us see the consequences. Suppose $(\mathbf{U}, \phi)$ is a solution of one of the reduced problems for $p^0 = p_n^0 = 0$. Then if we determine p by the appropriate formula, $(\mathbf{U}, p)$ will be a solution of (P) for p^0, p_n^0; hence by Remark 1.1 $\mathbf{U} \equiv 0$ in Ω and $p \equiv 0$ in Ω^+. For Method 1 we conclude that $\mathcal{S}[\phi] \equiv 0$ in Ω^+. By Lemma 2.1. (i) we can conclude $\phi \equiv 0$ and uniqueness for $(\tilde{P}_1)$, if and only if $\omega^2/c_0^2 \neq -\lambda_n^D$, where $\{\lambda_n^D\}$ are the eigen-values for $L_\omega v = 0$ in Ω with Dirichlet conditions. For Method 2 we have, by (2.7), $\mathcal{D}[\phi] \equiv 0$ in Ω^+ and we will have uniqueness only if $\omega^2/c^2 \neq -\lambda_n^N$, where $\{\lambda_n^N\}$ are the eigen-values for Neumann conditions. For Method 3 we will have $\mathcal{S}[\phi] + \alpha\mathcal{D}[\phi] \equiv 0$ in Ω^+; hence $\phi = 0$. Thus we have the following.

THEOREM 2.1. *If* $\operatorname{Im} \alpha \neq 0$ $(\tilde{P})$ *has at most one solution.*

In the next two sections we confine our attention to $(\tilde{P})$. We remark that the results obtained will be true for Methods 1 and 2 and the others in [1] as long as ω is not equal to a member of the appropriate bad sequence.

Proof of Lemma 2.1. Suppose $\mathcal{S}[\phi] \equiv 0$ in Ω^+. Put $v = \mathcal{S}[\phi]$. Then v is defined and continuous in all space, and $L_\omega v = 0$ in $\Omega \cup \Omega^+$. Further, we have $v^+ = v_n^+ = 0$. Thus v is a solution of the problem $L_\omega v = 0$ in Ω, $v^- = 0$. From (2.3c) we have then that $\phi = v_n^+ - v_n^- = v_n^-$. Conversely suppose $L_\omega v = 0$ in Ω with $v^- = 0$. Then, by (2.5) $v = -\mathcal{S}[v_n^-]$. Now consider $v = \mathcal{S}[v_n^-]$ in Ω^+. It is a solution of $L_\omega v = 0$ and vanishes on Γ; hence it is identically zero by uniqueness for the exterior problem. Thus $v^- = 0$ and $\phi = -v_n^-$ satisfies $S[\phi] = 0$. A similar argument proves the result for $\mathcal{D}$. Suppose now that $\mathcal{S}[\phi] + \alpha\mathcal{D}[\phi] \equiv 0$ in Ω^+ and put $w = \mathcal{S}[\phi] + \alpha\mathcal{D}[\phi]$. Then

$$v^+ = S[\phi] - \frac{\alpha}{2}\phi + \alpha D[\phi] = 0,$$

$$v^- = S[\phi] + \frac{\alpha}{2}\phi + \alpha D[\phi] = \alpha\phi,$$

$$v_n^+ = \frac{1}{2}\phi + N[\phi] + \alpha M[\phi] = 0,$$

$$v_n^- = -\frac{1}{2}\phi + N[\phi] + \alpha M[\phi] = -\phi.$$

Then by Green's theorem,

$$\int_\Omega \left(|\nabla v|^2 - \frac{\omega^2}{c_0^2}|v|^2\right) dx - \int_\Gamma v^- v_n^- \, ds$$
$$= \int_\Omega \left(|\nabla v|^2 - \frac{\omega^2}{c_0^2}|v|^2\right) dx + \alpha \int_\Gamma |\phi|^2 \, ds = 0;$$

Hence $\phi \equiv 0$. (This is essentially a proof from [3].)

III. VARIATIONAL PROBLEM

We give a variational formulation for $(\tilde{P})$. We multiply the differential equation by a test function $\overline{\mathbf{V}}$, integrate over Ω, apply Green's theorem, and use the first boundary condition to evaluate the boundary integral. Then we multiply the second boundary condition by a test function $\overline{\psi}$ and integrate over Γ. We introduce the following notation:

$$A(\mathbf{U}, \mathbf{V}) = \int_\Omega (\boldsymbol{\sigma}[\mathbf{U}] : \nabla\overline{\mathbf{V}} - \rho_0\omega^2 \mathbf{U}\overline{\mathbf{V}}) \, dx,$$

$$B(\phi, \mathbf{V}) = \int_\Gamma \left(S[\phi] - \frac{\alpha}{2}\phi + \alpha D[\phi]\right) \overline{\mathbf{V}} \cdot \mathbf{n} \, ds \tag{3.1a}$$

$$C(\mathbf{U}, \psi) = -\int_\Gamma \mathbf{U}^- \cdot \mathbf{n}\overline{\psi} \, ds,$$

$$D(\phi, \psi) = \frac{1}{\rho_0\omega^2} \int_\Gamma \left(\frac{1}{2}\phi + N[\phi] + \alpha M[\phi]\right) \overline{\psi} \, ds \tag{3.1b}$$

$$\mathcal{F}(\mathbf{V}^- \cdot \mathbf{n}) = -\int_\Gamma p^0 \overline{\mathbf{V}}^- \cdot \mathbf{n} \, ds,$$

$$\mathcal{G}(\psi) = -\frac{1}{\rho_0\omega^2} \int_\Gamma p_n^0 \overline{\psi} \, ds. \tag{3.1c}$$

We will seek generalized solutions of $(\tilde{P})$ in which $(\mathbf{U}, \phi) \in \mathcal{H} = (H_1(\Omega))^3 \times H_{1/2}(\Gamma)$ and we will require that the traces,

$$p^0, p_n^0 \in H_{1/2}(\Gamma). \tag{3.2}$$

Then by (2.4), we see that all terms in (3.1) will be meaningful for $(\mathbf{V}, \psi) \in \mathcal{H}$, provided we interpret $\int_\Gamma M[\phi]\overline{\psi} \, ds$ as the duality pairing $\langle M[\phi], \psi \rangle$ between $H_{1/2}(\Gamma)$ and $H_{-1/2}(\Gamma)$. Our variational problem then is to find $(\mathbf{U}, \phi) \in \mathcal{H}$ such that

$$A(\mathbf{U}, \mathbf{V}) + B(\phi, \mathbf{V}) = \mathcal{F}(\mathbf{V}),$$
$$C(\mathbf{U}, \psi) + D(\phi, \psi) = \mathcal{G}(\psi) \quad \forall (\mathbf{V}, \psi) \in \mathcal{H}. \tag{$V\tilde{P}$}$$

Remark 3.1. Note the difference in the regularity. In Method 1 $\phi \in H_{-1/2}(\Gamma)$ and the corresponding variational problem is on $H_1(\Omega)^3 \times H_{-1/2}(\Gamma)$. In Method 3 $\phi \in H_{1/2}(\Gamma)$ and the variational problem is on $H_1(\Omega)^3 \times H_{1/2}(\Gamma)$.

Our first result to be proved in the next section is this theorem.

THEOREM 3.1. $(V\tilde{P})$ *has a unique solution for any* ω *and any* p^0, p_n^0.

Although we will not carry out the proof, the following regularity result can be obtained. (See Remark 4.1 for the idea of this argument.)

THEOREM 3.2. *If* $p^0, p_n^0 \in H_{1/2+k}(\Gamma)$ *one has* $\mathbf{U} \in H_{2+k}(\Omega)$ *and* $\phi \in H_{1/2+k}(\Gamma)$.

For $k \geq 2$ it will follow that $(\mathbf{U}, \phi)$ is a classical solution of $(\tilde{P})$.

One obtains approximate problems by choosing families W^h, S^h of finite dimensional subspaces of $H_1(\Omega)$ and $H_{1/2}(\Gamma)$, and seeking $(\mathbf{U}^h, \phi^h) \in \mathcal{H}^h = W^h \times S^h$ such that

$$A(\mathbf{U}^h, \mathbf{V}^h) + B(\phi^h, \mathbf{V}^h) = \mathcal{F}(\mathbf{V}^h), \tag{$AV\tilde{P}$}$$
$$C(\mathbf{U}^h, \psi^h) + D(\phi^h, \psi^h) = \mathcal{G}(\psi^h), \quad \forall (\mathbf{V}^h, \psi^h) \in \mathcal{H}^h.$$

We will assume that the spaces W^h and S^h have the approximation property:

Given $\varepsilon > 0$ there is an h_0 such that for any $h < h_0$ and any $(\mathbf{U}, \phi) \in \mathcal{H}$ there is a $(\mathbf{U}^h, \psi^h) \in \mathcal{H}^h$ such that

$$\|(\mathbf{U}, \phi) - (\mathbf{U}^h, \psi^h)\|_{\mathcal{H}} \le \varepsilon.$$

Under this assumption we will establish the following optimal convergence result.

THEOREM 3.3. *Let* $(\mathbf{U}, \phi)$ *be the solution of* $(\tilde{P})$. *Then there is an* $h_0 > 0$ *and a* $\gamma > 0$ *such that for any* $h < h_0$:

(i) $(AV\tilde{P})$ *has a unique solution* $(\mathbf{U}^h, \psi^h)$

(ii) $\|(\mathbf{U}, \phi) - (\mathbf{U}^h, \phi^h)\|_{\mathcal{H}} \le \gamma \|(\mathbf{U}, \phi) - (\mathbf{V}^h, \psi^h)\|_{\mathcal{H}}$ $\forall (\mathbf{V}^h, \psi^h) \in \mathcal{H}^h$.

It follows that $(\mathbf{U}^h, \phi^h) \to (\mathbf{U}, \phi)$ in $\mathcal{H}$. The rather technical arguments of the next section amount to showing that $(V\tilde{P})$ is a compact perturbation of a coercive problem; hence it is a Riesz–Schauder system for which uniqueness implies existence. Thus the failure of uniqueness produces a failure of existence.

IV. EXISTENCE AND CONVERGENCE

A key to the work in this section is that the forms A and D of (3.1) are nearly coercive. For A this follows from Korn's second inequality which states that there are constants $k_1 > 0$, $k_0 > 0$ such that

$$\int_\Omega \boldsymbol{\sigma}[\mathbf{U}]: \nabla\overline{\mathbf{U}}\, dx \ge k_1\|\mathbf{U}\|^2_{H_1(\Omega)} - k_0\|\mathbf{U}\|^2_{L_2(\Omega)}. \tag{4.1}$$

For D we need to compare M with the operator obtained if we replace K in (2.1) by $K_0(z) = -(4\pi z)^{-1}\exp(-z)$, the fundamental singularity for $\Delta v - v = 0$.[3] We write $K = K_0 + K_1$, $D = D_0 + D_1$, $M = M_0 + M_1$. Then there is a cancellation of singular terms so that one has

$$M_1: H_r(\Gamma) \to H_{r+1}(\Gamma). \tag{4.2}$$

For M_0 we have the following result.

LEMMA 4.1. *There is a constant* $m > 0$ *such that for any* $\phi \in H_{1/2}(\Gamma)$

$$\langle M_0\phi, \phi\rangle > m\|\phi\|^2_{H_{1/2}(\Gamma)}. \tag{4.3}$$

Proof. A complete proof is given in [6] but we can sketch the essential idea. Assume ϕ is smooth and put $v = \mathcal{D}_0[\phi]$. Then $v^- - v^+ = \phi$ and $v_n^+ = v_n^- = M_0[\phi]$. Apply Green's theorem to Ω and Ω^+ to obtain

$$0 = \int_\Omega (|\nabla v|^2 + |v|^2)\, dx - \int_\Gamma v_n^- v^-\, dS$$
$$= \int_{\Omega^+} (|\nabla v|^2 + |v|^2)\, dx + \int_\Gamma v_n^+ v^+\, dS.$$

Hence,

$$\int_\Gamma (v_n^- v^- - v_n^+ v^+)\, dS = \int_\Gamma M_0[\phi]\phi\, dS$$
$$= \int_\Omega (|\nabla v|^2 + |v|^2)\, dx + \int_{\Omega^+} (|\nabla v|^2 + |v|^2)\, dx.$$

This shows that M_0 is injective. Pseudo-differential operator theory (M_0 is an operator of order one) can then be invoked to show that M_0 is also surjective and (4.3) follows from the open mapping theorem.

We make the following decomposition for problem $(V\tilde{P})$. We set

$$A = A_0 + A_1, \quad D = D_0 + D_1 \tag{4.4}$$

with

$$A_0(\mathbf{U}, \mathbf{V}) = \int_\Omega (\boldsymbol{\sigma}[\mathbf{U}]: \nabla\overline{\mathbf{V}} + k_0\mathbf{U}\cdot\overline{\mathbf{V}})\, dx$$
$$= \langle \tilde{A}_0(\mathbf{U}), \mathbf{V}\rangle \tag{4.5a}$$

$$\langle M_0(\phi), \psi\rangle = \langle \tilde{M}_0(\phi), \psi\rangle. \tag{4.5b}$$

Here $\tilde{A}_0$ and $\tilde{M}_0$ are bounded, invertible, linear maps from $H_1(\Omega)^3$ and $H_{1/2}(\Gamma)$ to their duals, $(H_1(\Omega)^3)'$ and $H_{1/2}(\Gamma))' = H_{-1/2}(\Gamma)$. First we consider the variational problem,

$$A_0(\mathbf{U}, \mathbf{V}) = \mathcal{F}(\mathbf{V}), \quad M_0(\phi, \psi) = \mathcal{L}(\psi) \tag{4.6}$$

(that is, $\tilde{A}_0(U) = \mathcal{F}$ and $\tilde{M}_0\phi = \mathcal{L}$) for bounded linear functionals $\mathcal{F}$ and $\mathcal{L}$ on $H_1(\Omega)^3$ and $H_{1/2}(\Gamma)$. The invertibility of $\tilde{A}_0$ and $\tilde{M}_0$ yields the following result.

LEMMA 4.2. *There exists a unique solution of* (4.6),

$$(\mathbf{U}, \phi) = (\tilde{A}_0^{-1}(\mathcal{F}), \tilde{M}_0^{-1}(\mathcal{L})), \tag{4.7}$$

where $\tilde{A}_0^{-1}$ *and* $\tilde{M}_0^{-1}$ *are bounded linear maps.*

Next we write, by (3.1), (4.4), and (4.5),

$$A_1(\mathbf{U}, \mathbf{V}) = -\int_\Omega (k_0 + \rho\omega^2)\mathbf{U}\cdot\overline{\mathbf{V}}\, dx \equiv \langle \tilde{A}_1(\mathbf{U}), \mathbf{V}\rangle \tag{4.8a}$$

$$B(\phi, \mathbf{V}) = \int_\Gamma \left(S[\phi] - \frac{\alpha}{2}\phi + \alpha D[\phi]\right)\overline{\mathbf{V}}^-\cdot\mathbf{n}\, ds$$
$$= \langle \tilde{B}(\phi), \mathbf{V}\rangle \tag{4.8b}$$

[3] In two dimensions $K_0 = -i/4 \ln z$.

$$C(\mathbf{U}, \psi) = \int_\Gamma \mathbf{U}^- \cdot \mathbf{n}\bar{\psi}\, ds = \langle \tilde{C}(\mathbf{U}), \psi\rangle \tag{4.8c}$$

$$D_1(\phi, \psi) = \frac{1}{\rho_0\omega^2}\int_\Gamma \left(\frac{1}{2}\phi + N[\phi] + \alpha M_1[\phi]\right)\bar{\psi}\, ds = \langle \tilde{D}_1(\phi), \psi\rangle. \tag{4.8d}$$

LEMMA 4.3. *$\tilde{A}_1$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}_1$ are compact.*

Proof. $\tilde{A}_1$ is the composition of the injection of $H_1(\Omega)^3$ into $L_2(\Omega)^3$, which is compact and the embedding of $L_2(\Omega)^3$ into $(H_1(\Omega)^3)'$ which is continuous. B is the composition of the map ϕ into $S[\phi] - \frac{1}{2}\alpha\phi + \alpha D[\phi]$ from $H_{1/2}(\Gamma)$ into itself, the natural embedding of $H_{1/2}(\Gamma)$ into $H_{-1/2}(\Gamma)$, which is compact, and the transpose of the operator taking $H_1(\Omega)^3$ into the normal component of the trace. C composes the normal component of the trace operator with the injection of $H_{1/2}(\Gamma)$ into $H_{-1/2}(\Gamma)$. Finally D_1 composes a map from $H_{1/2}(\Gamma)$ to itself with the injection of $H_{1/2}(\Gamma)$ into $H_{-1/2}(\Gamma)$.

Proof of Theorem 3.2. On the basis of the calculations above we see that $(V\tilde{P})$ is equivalent to the equations

$$\mathbf{U} = (\tilde{A}_0^{-1}\tilde{A}_1 + \tilde{A}_0^{-1}\tilde{B})\mathbf{U} + \tilde{A}_0^{-1}\mathcal{F} \quad \text{on } (H_1(\Omega))^3) \tag{4.9a}$$

$$\phi = (\tilde{M}_0^{-1}\tilde{C} + \tilde{M}_0^{-1}\tilde{D}_1)\phi + \tilde{M}_0^{-1}\mathcal{G} \quad \text{on } H_{1/2}(\Gamma), \tag{4.9b}$$

where the operators on the right are compact. Hence, to establish existence it suffices to show that the only solution for $\mathcal{F} = 0$, $\mathcal{G} = 0$ is $\mathbf{U} = 0$, $\phi = 0$. But if $(\mathbf{U}, \phi)$ is such a solution $\mathbf{U}$ and $p = \mathcal{S}[\phi] + \alpha\mathcal{D}[\phi]$ will yield a solution of $\tilde{P}$ for $p^0 = p_n^0 = 0$; hence $\mathbf{U} = 0$ and $\phi = 0$.

Remark 4.1. To make this argument completely rigorous one needs to show that a solution of the homogeneous equation (4.9) has sufficient regularity so that the uniqueness theorem, Theorem 2.1 applies. Let us sketch just a little of this argument. If $(\mathbf{U}, \phi)$ is a solution of the homogeneous equation (4.9) then we see that $\mathbf{U}$ is a generalized solution of div $\boldsymbol{\sigma}[\mathbf{U}] - k_0\mathbf{U} = -(k_0 + \rho\omega^2)\mathbf{U}) \equiv \boldsymbol{\chi}$ with the traction condition $\boldsymbol{\sigma}(\mathbf{U})[\mathbf{n}] = -(S[\phi] - (\alpha/2)\phi + \alpha D[\phi] + p^0)\mathbf{n} \equiv \mu\mathbf{n}$. We have $\boldsymbol{\chi} \in L_2(\Omega)$ and $\mu \in H_{1/2}(\Gamma)$ and standard elliptic theory then gives $\mathbf{U} \in H_2(\Omega)^3$. Then $\mathbf{U}^- \cdot \mathbf{n} \in H_{3/2}(\Gamma)$ and the equation $\frac{1}{2}\phi + N[\phi] + \alpha M[\phi] = \rho_0\omega^2\mathbf{U}^- \cdot \mathbf{n}$ will give $\phi \in H_{3/2}(\Gamma)$. One can now continue by a boot-strapping argument to show that $\mathbf{U}$ and ϕ have an arbitrary degree of smoothness if Γ is smooth.

Proof of Theorem 3.3. This result follows from the same two facts, coercivity and compactness, as Theorem 3.2. This is a known technique; see, for instance, [1 or 9]. We sketch the essential idea. One first defines a Galerkin operator G_0^h: $\mathcal{H} \to \mathcal{H}^h$ by the formula $G_0^h(\mathbf{U}, \phi) = (\tilde{\mathbf{U}}^h, \tilde{\phi}^h)$, where

$$A_0(\tilde{\mathbf{U}}^h, \mathbf{V}^h) = A_0(\mathbf{U}, \mathbf{V}^h) \quad \forall \mathbf{V}^h \in W^h \tag{4.10a}$$

$$M_0(\tilde{\phi}^h, \psi^h) = M_0(\phi, \psi^h) \quad \forall \psi^h \in S^h. \tag{4.10b}$$

Lemma 4.2 shows that G_0^h exists and one can show that

$$\|G_0^h(\mathbf{U}, \phi) - (\mathbf{U}, \phi)\|_{\mathcal{H}} \leq K\|(\mathbf{U}, \phi)\|_{\mathcal{H}}.$$

We write $\mathcal{K}(\mathbf{U}, \phi)$ for the map from $\mathcal{H}$ into $\mathcal{H}'$ defined by (4.5). Then we define an operator $\mathcal{L}^h$: $\mathcal{H} \to \mathcal{H}$ by the formula

$$\mathcal{L}^h = \mathcal{T}\mathcal{K}(I - G_0^h),$$

where $\mathcal{T}$ is the mapping from $\mathcal{H} \to \mathcal{H}$ defined by the compact operators in Lemma 4.3. The compactness of $\mathcal{K}$ can then be shown to imply that $\|\mathcal{L}^h\| \to 0$ as $h \to 0$, provided the spaces $\mathcal{H}^h$ approximate $\mathcal{H}$. Finally one can then define a Galerkin operator $G^h = G_0^h(I - \mathcal{L}^h)^{-1}$ for the perturbed system $(V\tilde{P})$.

V. SYMMETRIC FORMULATION

Before discussing the numerical experiments, in this section we give a symmetric version of our procedure, which is the one that was used in the actual numerical implementation. We recall that our stable procedure is indirect since it is based on the representation (I) for p in Ω^+ in terms of a combination of simple and double layers with density ϕ. Here we use the method proposed in [4], to generate a direct stable formulation. A combination with our previous method then yields the desired symmetric form.

METHOD 4. This starts with the same direct representation formula (2.5) for p in Ω^+ as method 2, $p = \mathcal{S}[p_n^+] - \mathcal{D}[p^+]$ in Ω^+, from which we obtain

$$(\tfrac{1}{2}p^+ - S[p_n^+] + D[p^+]) + \alpha(\tfrac{1}{2}p_n^+ - N[p_n^+] + M[p^+]) = 0 \tag{D}$$

with Im $\alpha \neq 0$, and we introduce $\lambda = \boldsymbol{\sigma}[\mathbf{U}^-](\mathbf{n}) \cdot \mathbf{n}$, as a new auxiliary variable. After inserting the transition conditions from (P) into (D) this yields the reduced problem

$$\begin{aligned} &\text{div}\, \boldsymbol{\sigma}[\mathbf{U}] + \rho\omega^2\mathbf{U} = 0 \quad \text{in } \Omega \\ &\boldsymbol{\sigma}[\mathbf{U}^-](\mathbf{n}) = \lambda\mathbf{n} \quad \text{on } \Gamma \\ &\rho_0\omega^2 S[\mathbf{U}^- \cdot \mathbf{n}] + \tfrac{1}{2}\lambda + D[\lambda] + \alpha(-\tfrac{1}{2}\rho_0\omega^2\mathbf{U}^- \cdot \mathbf{n} \\ &\quad + \rho_0\omega^2 N[\mathbf{U}^- \cdot \mathbf{n}] + M[\lambda]) \\ &= S[p_n^0] - D[p^0] + \alpha N[p_n^0] - \alpha M[p^0] \\ &= -(p^0 + \alpha p_n^0) \quad \text{on } \Gamma \end{aligned} \tag{$\tilde{P}_D$}$$

for $(\mathbf{U}, \lambda)$, p is then determined in Ω^+ by

$$p = \rho_0\omega^2\mathcal{S}[\mathbf{U}^- \cdot \mathbf{n}] - \mathcal{D}[-\lambda - p^0]. \tag{5.1}$$

Remark 5.1. We have simplified the right side of the last equation in $(\tilde{P}_D)$ by using the representation formula (2.5a) for p^o in Ω.

A symmetric formulation is now obtained by combining $(\tilde{P})$ and $(\tilde{P}_D)$ as follows.

METHOD 5. Here the problem is to solve for (U, λ, ϕ) such that

$$\operatorname{div} \boldsymbol{\sigma}[\mathbf{U}] + \rho\omega^2 \mathbf{U} = 0 \quad \text{in } \Omega$$

$$\boldsymbol{\sigma}[\mathbf{U}^-](\mathbf{n}) = \frac{1}{2}\left(\lambda - S[\phi] + \frac{\alpha}{2}\phi - \alpha D[\phi] - p^0\right)\mathbf{n} \quad \text{on } \Gamma$$

$$-\frac{1}{2}\mathbf{U}^- \cdot \mathbf{n} + \frac{1}{2\rho_0\omega^2}\left(\frac{1}{2}\phi + N[\phi] + \alpha M[\phi] + p_n^0\right) = 0 \quad \text{on } \Gamma$$

$$\frac{1}{2}S[\mathbf{U}^- \cdot \mathbf{n}] + \frac{1}{2\rho_0\omega^2}\left(\frac{1}{2}\lambda + D[\lambda] + \alpha M[\lambda]\right) \qquad (\tilde{P}_S)$$

$$+ \frac{\alpha}{2}\left(-\frac{1}{2}\mathbf{U}^- \cdot \mathbf{n} + N[\mathbf{U}^- \cdot \mathbf{n}]\right)$$

$$= \frac{1}{2\rho_0\omega^2}(-p^0 - \alpha p_n^0) \quad \text{on } \Gamma.$$

p is then determined in Ω^+ *by either* (I) *or* (5.1).

The same results obtained for Method 3 hold for Methods 4 and 5. They will be true also for the limiting case corresponding to vanishing α in $(\tilde{P}_D)$ and $(\tilde{P}_S)$ as long as $\omega^2/c_0^2 \neq -\lambda_n^D$.

Next we give a variational formulation for $(\tilde{P}_S)$. This will serve as a basis for the numerical results for the deformable scatterer to be presented subsequently. We multiply the differential equation by a test function $\overline{\mathbf{V}}$, integrate over Ω, apply Green's theorem, and use the first boundary condition to evaluate the boundary integral. Then we multiply the second and third boundary conditions by test functions $\overline{\psi}$ and $\overline{\chi}$, respectively, and integrate over Γ. Using the notation (3.1) this process leads to

$$A(\mathbf{U}, \mathbf{V}) + \tfrac{1}{2}C(\overline{\mathbf{V}}, \overline{\lambda}) - \tfrac{1}{2}B(\phi, \mathbf{V}) = \tfrac{1}{2}\mathcal{F}(\mathbf{V}^- \cdot \mathbf{n})$$

$$\tfrac{1}{2}C(\mathbf{U}, \psi) + \tfrac{1}{2}D(\phi, \psi) = \tfrac{1}{2}\mathcal{G}(\psi) \qquad (V\tilde{P}_S)$$

$$-\tfrac{1}{2}B(\overline{\chi}, \overline{\mathbf{U}}) + \tfrac{1}{2}D(\overline{\chi}, \overline{\lambda}) = \tfrac{1}{2}\mathcal{F}(\chi) + \tfrac{1}{2}\mathcal{G}(\chi).$$

Our variational problem then is to find $(\mathbf{U}, \phi, \lambda) \in \mathcal{H}^* = (H_1(\Omega))^3 \times (H_{1/2}(\Gamma)) \times (H_{1/2}(\Gamma))$ *such that* $(V\tilde{P}_S)$ *holds for all* $(\mathbf{V}, \chi, \psi) \in \mathcal{H}$. *We remark that* $(V\tilde{P}_{\mathcal{S}})$ *may be written symbolically as*

$$\mathcal{A}((\mathbf{U}, \phi, \lambda), (\mathbf{V}, \chi, \psi)) = \mathcal{T}((\mathbf{V}, \chi, \psi))$$

and that the trilinear form $\mathcal{A}$ has the symmetry property

$$\mathcal{A}((\mathbf{U}, \phi, \lambda), (\mathbf{V}, \chi, \psi)) = \mathcal{A}((\overline{\mathbf{V}}, \overline{\chi}, \overline{\psi}), (\mathbf{U}, \phi, \lambda)).$$

In the finite-dimensional approximation this means that the matrices will be symmetric if we use real basis elements.

The approximate problem $(AV\tilde{P}_S)$ corresponding to $(V\tilde{P}_S)$ consists in seeking $(\mathbf{U}^h, \phi^h, \lambda^h) \in \mathcal{H}_1^h = \mathcal{W}^h \times \mathcal{S}^h \times \mathcal{S}^h$ such that

$$\mathcal{A}((\mathbf{U}^h, \phi^h, \lambda^h), (\mathbf{V}^h, \chi^h, \psi^h)) = \mathcal{T}((\mathbf{V}^h, \chi^h, \psi^h)) \quad \forall (\mathbf{V}^h, \chi^h, \psi^h) \in \mathcal{H}_1^h. \qquad (AV\tilde{P}_S)$$

Let us study the structure of these equations. Suppose that our basis for $\mathcal{W}^h$ has the form $\boldsymbol{\beta}_1^h, \ldots, \boldsymbol{\beta}_{N_\Omega^h}^h, \boldsymbol{\gamma}_1^h, \ldots, \boldsymbol{\gamma}_{N_\Gamma^h}^h$, with $\boldsymbol{\beta}_j^h = 0$ on Γ, and let $\sigma_1^h, \ldots, \sigma_m^h$, be a basis for $\mathcal{S}^h$. Then $(\mathbf{U}^h, \phi^h, \lambda^h)$ has a corresponding decomposition into $\mathbf{U}_\Omega^h$, $\mathbf{U}_\Gamma^h$, ϕ^h, λ^h, and $(AV\tilde{P}_S)$ assumes the form

$$\begin{bmatrix} \mathbf{A}_{\Omega\Omega}^h & \mathbf{A}_{\Omega\Gamma}^h & 0 & 0 \\ (\mathbf{A}_{\Omega\Gamma}^h)^{\mathrm{T}} & \mathbf{A}_{\Gamma\Gamma}^h & \tfrac{1}{2}(C^h)^{\mathrm{T}} & -\tfrac{1}{2}\mathbf{B}^h \\ 0 & \tfrac{1}{2}C^h & 0 & \tfrac{1}{2}\mathbf{D}^h \\ 0 & -\tfrac{1}{2}(\mathbf{B}^h)^{\mathrm{T}} & \tfrac{1}{2}(\mathbf{D}^h)^{\mathrm{T}} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_\Omega^h \\ \mathbf{U}_\Gamma^h \\ \boldsymbol{\lambda}^h \\ \boldsymbol{\phi}^h \end{Bmatrix} = \begin{Bmatrix} 0 \\ \tfrac{1}{2}\mathcal{F}_\Gamma^h \\ \tfrac{1}{2}\mathcal{F}_\lambda^h \\ \tfrac{1}{2}\mathcal{F}_\phi^h \end{Bmatrix}. \qquad (5.2)$$

The elements of the various matrices $\mathbf{A}^h$, $\mathbf{B}^h$, $\mathbf{C}^h$, and $\mathbf{D}^h$, are expressed in terms of the corresponding bilinear forms in $(V\tilde{P}_S)$; e.g., $(\mathbf{D}^h)_{ij} = D(\sigma_i^h, \sigma_j^h)$. Similarly, the forcing functions on the right side of (5.2) are related to the right sides of $(V\tilde{P}_S)$, and are functions of the incoming field p^0 and its derivative p_n^0, on Γ.

Remark 5.2. An advantage of a symmetric formulation, *such as the one in* (5.2), *is that the algebraic problem can be* solved efficiently by iterative techniques. The procedure is also *amenable to condensation. For instance, if one is concerned* primarily with the interior region, one may eliminate $\boldsymbol{\lambda}^h$ and $\boldsymbol{\phi}^h$ using the last two equations in (5.2) and substitute into the second equation. (5.2) then becomes

$$\begin{bmatrix} \mathbf{A}_{\Omega\Omega}^h & \mathbf{A}_{\Omega\Gamma}^h \\ (\mathbf{A}_{\Omega\Gamma}^h)^{\mathrm{T}} & \mathbf{A}_{\Gamma\Gamma}^h + \mathbf{K}_{\Gamma\Gamma}^h \end{bmatrix} \begin{Bmatrix} \mathbf{U}_\Omega^h \\ \mathbf{U}_\Gamma^h \end{Bmatrix} = \begin{Bmatrix} 0 \\ \mathcal{G}_\Gamma^h \end{Bmatrix}, \qquad (5.3)$$

where

$$\mathbf{K}_{\Gamma\Gamma}^h = -\tfrac{1}{2}(\mathbf{B}^h(\mathbf{D}^h)^{-1}\mathbf{C}^h + (\mathbf{B}^h(\mathbf{D}^h)^{-1}\mathbf{C}^h)^{\mathrm{T}})$$

and

$$\mathcal{G}_\Gamma^h = \tfrac{1}{2}(\mathcal{F}_\Gamma^h + (\mathbf{C}^h)^{\mathrm{T}}(\mathbf{D}^h)^{-\mathrm{T}}\mathcal{F}_\phi^h - \mathbf{B}^h(\mathbf{D}^h)^{-1}\mathcal{F}_\lambda^h).$$

The matrix $\mathbf{K}_{\Gamma\Gamma}^h$ is symmetric and full and represents the impedance of the exterior region Ω^+; it constitutes, in effect, a discretized nonlocal absorbing boundary. $\mathscr{G}_\Gamma^h$ represents the corresponding effective forcing function.

VI. NUMERICAL EXPERIMENTS

In order to assess the stability of our procedure we consider the two-dimensional scattering problem for a circular cylinder of radius a to an incident plane wave of amplitude P^0. The simple problem of a rigid scatterer is solved first. This means the exterior Neumann problem for $L_\omega p = 0$ with $p_n = -p_n^0$. Subsequently, we treat the interface problem. The axis of the cylinder is kept fixed in both cases, and piecewise quadratic isoparametric elements are used in all the calculations. Since the main objective of the numerical experiments with the rigid scatterer is to investigate how various methods perform near singular frequencies, we use a sufficiently large number of finite elements so that any major discrepancy with the exact solution will be due to the lack of stability of the method itself. Thus 32 finite elements were used on Γ to approximate both the pressure and the density of the potential layers. The numerical treatment of the duality pairing $\int_\Gamma M[\phi]\overline{\psi}\, d\Gamma$ that contains the hypersingular operator M is becoming standard and uses the identity

$$\int_\Gamma \overline{\psi} M[\phi]\, ds = k^2 \int_\Gamma S[\phi \mathbf{n}] \cdot (\overline{\psi}\mathbf{n})\, ds - \int_\Gamma S[\mathbf{n} \times \nabla\phi] \cdot (\mathbf{n} \times \nabla\overline{\psi})\, ds$$

in which $k = \omega/c_0$ is the wave number. All the other operators in our formulation are standard; thus, they require no further explanation, except to note that three-point Gauss–Legendre integration formulas were used after subtracting off singularities. (In $\mathbf{R}^2$ these singularities are only logarithmic.)

In Figs. 2a and 2b we used the discretized version of the variational formulations corresponding to the generalized indirect and direct representations given in Methods 3 and 2 respectively. That is, the results plotted in Fig. 2a for the total pressure at point A directly on the scatterer (Fig. 1) are obtained from the solutions for $\phi^h \in S^h$ of

$$\langle \tfrac{1}{2}\phi^h + N[\phi^h] + \alpha M[\phi^h], \psi^h \rangle = -\langle p_n^0, \psi^h \rangle \quad \forall \psi^h \in S^h, \quad (5.4)$$

with $\alpha = 0$, while those in Fig. 2b come from the solution $p^h \in S^h$ of

$$\langle M[p^h], \psi^h \rangle = \langle -\tfrac{1}{2}p_n^0 + N[p_n^0], \psi^h \rangle \quad \forall \psi^h \in S^h. \quad (5.5)$$

Boundary integral equations are usually solved by collocation, rather than through variational procedures such as those described in the preceding two equations. To compare the corresponding results, Fig. 2c shows again the pressure at point A, but this time it is calculated by solving the integral equation

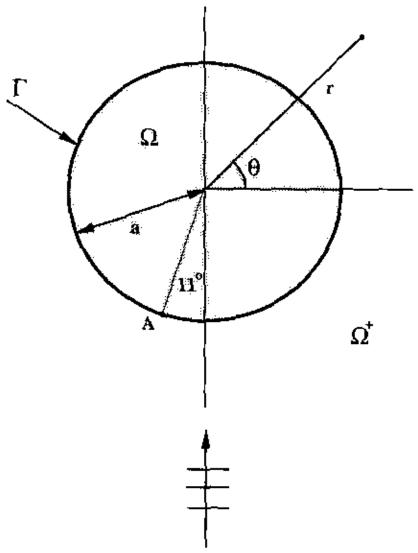

FIG. 1. Model of inhomogenous elastic circular cylinder immersed in a compressible inviscid fluid medium.

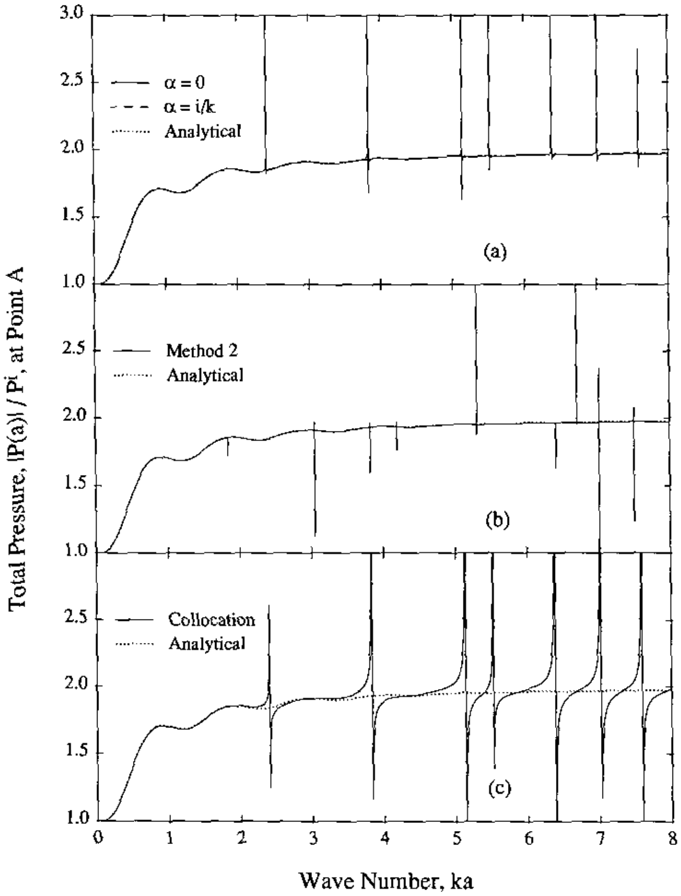

FIG. 2. Normalized amplitude of total pressure at point A on the surface of circular rigid scatterer: (a) Eq. (5.4); (b) Eq. (5.5); (c) Eq. (5.6).

$$\tfrac{1}{2}p^+ + D[p^+] = -S[p_n^0] \qquad (5.6)$$

directly by collocation. Note that the integral operator in (5.4) with $\alpha = 0$, (5.5), and (5.6) are non-stable, while those of (5.4) are stable if $\text{Im}\ \alpha \neq 0$.

It is important to emphasize that in contrast with the collocation method the variational procedures require that certain double integrals be evaluated. We found, however, that the total computational effort for solving (5.5) and (5.6) was comparable, since fewer Gaussian integration points were required (an average of four per integral in (5.5) versus eight in (5.6) per element).

The collocation solution is compared in Fig. 2c against the corresponding well-known exact solution. As expected, the approximate solution fails at the critical frequencies corresponding to the eigen-values λ_n^D,[4] as it should, since the method is not stable. However, it also performs poorly within a sizable interval around each critical frequency. As the wave number increases and the critical frequencies become clustered together, the solution is unreliable throughout the high frequency range.

The exact solution of the rigid scattering problem is also shown in Figs. 2a and 2b, together with the numerical solutions. Again, the solutions of (5.4) with $\alpha = 0$ and (5.5) fail at the critical frequencies, corresponding to the eigen-values λ_n^N and λ_n^D, respectively. What is surprising is that these non-stable solutions differ from the exact solution only when ω is very near the critical frequencies. The stable solution corresponding to (5.4) with $\alpha = i/k$ is indistinguishable from the exact solution, at all frequencies. These results confirm that the generalized indirect representation is valid for all frequencies, and, most importantly, they indicate that variational methods, even if non-stable, are preferable to collocation in the treatment of integral equations associated with wave propagation problems.

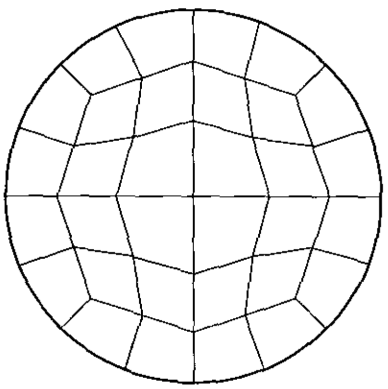

FIG. 3. Typical finite element mesh for numerical solution of scattering problem for elastic cylinder.

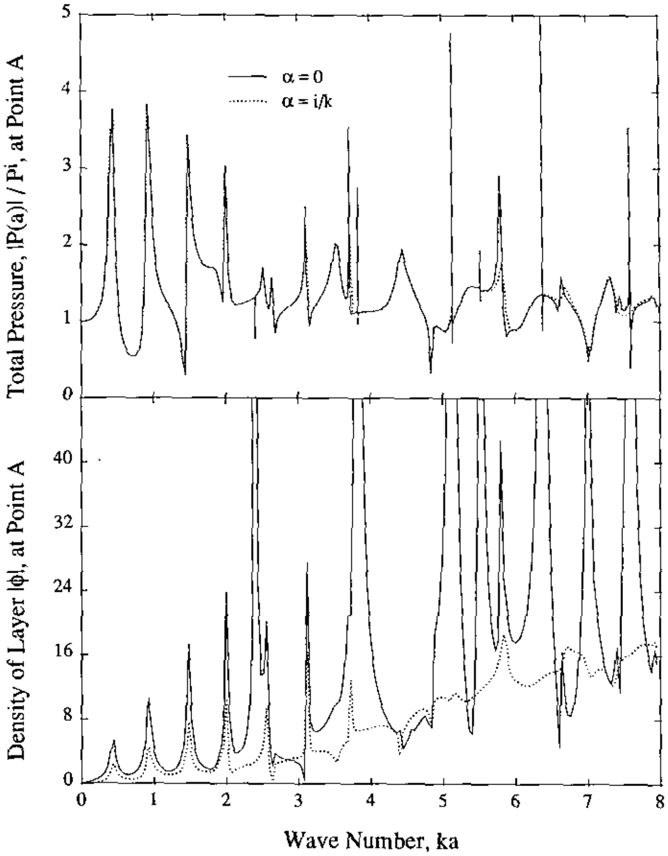

FIG. 4. Normalized amplitude of total pressure and amplitude of density of potential layers, at point A, for elastic scatterer ($\rho/\rho_0 = 1.1$, $c/c_0 = 0.576$, $\nu = 0.4$).

Our second example concerns the interface problem. We use (5.2) both with $\alpha = i/k$ (the stable case) and with $\alpha = 0$ (the non-stable case). Our theory guarantees that the former case is essentially exact for sufficient mesh refinement. Thus any differences in the two cases represent errors in the non-stable method.

The solution of (5.2) provides directly the displacement $\mathbf{U}$, the traction $\lambda = \boldsymbol{\sigma}[\mathbf{U}](\mathbf{n}) \cdot \mathbf{n}$,[5] and the density ϕ for the exterior pressure representation.

We consider a homogeneous, isotropic, elastic scatterer made of hard rubber. It has a constant Poison's ratio 0.4, mass density and shear wave velocity given in dimensionless form by $\rho/\rho_0 = 1.1$ and $c/c_0 = 0.576$. The scatterer is submerged in sea water and is subjected to a plane incident wave of amplitude P^0 as before. All the calculations are performed using (5.2) (once with $\alpha = i/k$ and once with $\alpha = 0$).

A typical mesh is indicated in Fig. 3. The response at point A on the surface of the scatterer (Fig. 1) is shown on Fig. 4. The amplitude of the total pressure[6] appears on Fig. 4a, while

[4] Critical frequencies are determinated as zeroes of Bessel functions or their derivatives.

[5] Note that λ also equals the total pressure on the surface.

[6] Evaluated from $P(a) = \lambda$.

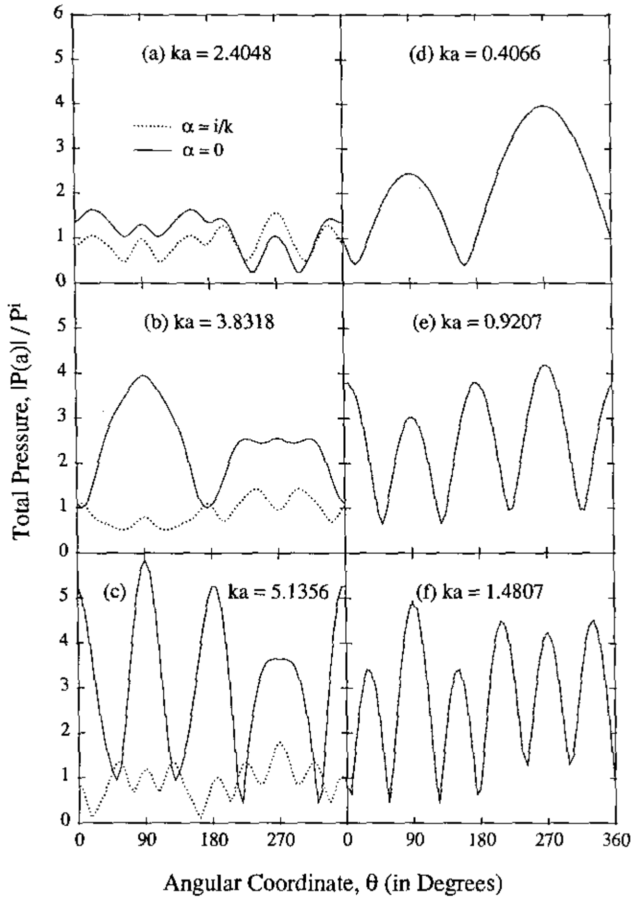

FIG. 5. Normalized amplitude of total pressure on the surface of elastic scatterer ($\rho/\rho_0 = 1.1$, $c/c_0 = 0.576$, $\nu = 0.4$) for various wave numbers (a, b, c) correspond to critical frequencies, whereas (d, e, f) correspond to the first three resonant frequencies of the fluid–structure system.)

the density of the potential layers is shown on Fig. 4b. The solid lines correspond to $\alpha = 0$ and the dotted ones to $\alpha = i/k$. (We recall that for $\alpha = 0$, ϕ is the density of a single layer, whereas for $\alpha = i/k$ it represents the density of a combined single and double layer. Thus, it is natural that the corresponding curves are different.)

The response of the deformable scatterer is much more complicated than that of the rigid scatterer, due to the compliance of the elastic scatterer. In addition to the spurious resonances exhibited at the critical frequencies by the non-stable procedure, there are additional peaks for both $\alpha = 0$ and $\alpha = i/k$. These correspond to actual resonant frequencies of the structure–fluid system. The resonant values for the two methods essentially coincide, except at higher frequencies.

In order to examine in greater detail the performance of stable and non-stable solutions both at critical frequencies and at actual resonant frequencies, we present in Fig. 5 the distribution of total pressure on the surface of the scatterer, as a function of the angular coordinate θ (Fig. 1), for several specific wavenumbers. Figures 5a,b,c correspond to three critical wavenumbers, whereas Figs. 5d,e,f are for the first three resonant frequencies of the system. As expected, the results of the non-stable solutions exhibit large errors at the critical frequencies; at the resonant frequencies, however, they are indistinguishable from the corresponding stable solutions. Note that at resonance the pressure in the fluid is significantly amplified over the maximum value of 2 that would be observed for a rigid scatterer, due to the deformation of the scatterer. Note also the oscillation in the pressure at resonance, as the structure–fluid system is excited in its various "modes" of vibration. Thus, Figs. 5d,e,f correspond to the first, second, and third modes, which describe, respectively, 1, 2, and 3 cycles around the scatterer. The peak values of the response are bounded because of the energy radiated into the fluid.

As a further illustration of the performance of the two methods, Fig. 6 shows the total pressure within the fluid, analogously to Fig. 5, but at a radius $r = 1.5a$. To calculate these results we made use of (I). The total pressure away from the scatterer naturally is smaller than that on the scatterer, although the overall patterns remain the same. It is interesting that the difference between the non-stable and stable solutions is much

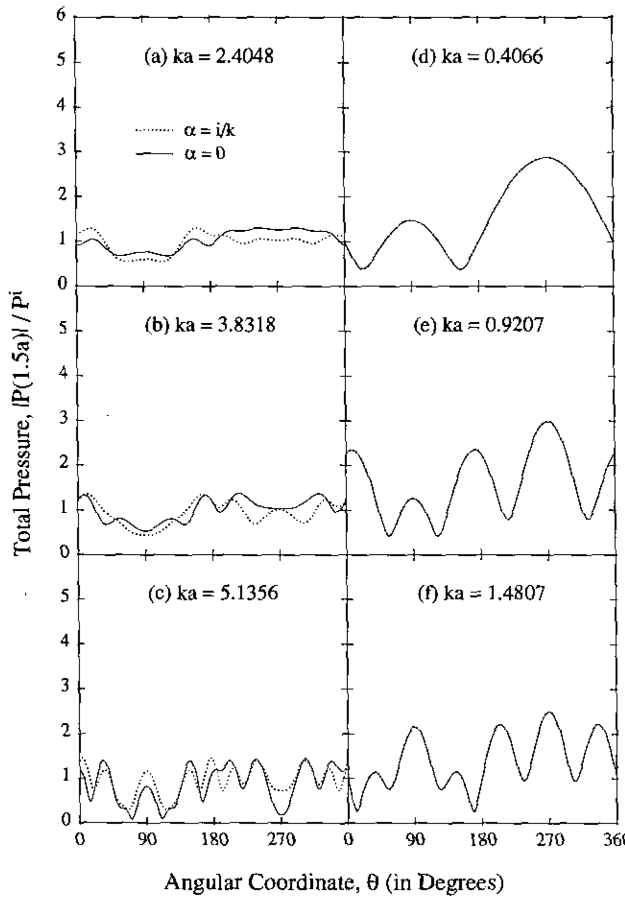

FIG. 6. Normalized amplitude of total pressure within the fluid on a circle of radius $1.5a$; same scatterer and wave numbers as in Fig. 5.

smaller away from the scatterer than directly on it. This is consistent with the well-known fact that far fewer elements are required on the boundary of a rigid scatterer to obtain a solution in the far field to within a prescribed error than directly on the scatterer, for any given method.

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